CRNOGORSKA AKADEMIJA NAUKA I UMJETNOSTI
GLASNIK ODJELJENJA PRIRODNIH NAUKA, 28, 2025.

ČERNOGORSKAYA AKADEMIYA NAUK I ISSKUSTV
GLASNIK OTDELENIYA ESTESTVENNYH NAUK, 28, 2025.

THE MONTENEGRIN ACADEMY OF SCIENCES AND ARTS
PROCEEDINGS OF THE SECTION OF NATURAL SCIENCES, 28, 2025

UDK 539.319

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DUALITY BETWEEN AZIMUTHAL SHEAR AND RADIAL LOADING OF A HOLLOW CIRCULAR CYLINDER AND RELATED PROBLEMS

Abstract

The Navier equation of equilibrium for the circumferential displacement $u_\theta = u_\theta(r)$ in a hollow circular cylinder subjected to azimuthal shear is of the same equidimensional type as the corresponding equation for the radial displacement $u_r = u_r(r)$ in the Lamé problem of radial loading of a hollow cylinder. The maximum shear stress in both problems varies as $1/r^2$, where r is the radial distance from the central axis of the cylinder, and is given by analogous expressions, which implies that the onset of plastic deformation is also defined by analogous expressions. Different stress functions for the Lamé problem are discussed in the context of a non-standard form of the compatibility condition, which yields a third-order differential equation for the Airy stress function, rather than the common fourth-order biharmonic differential equation. Two types of boundary conditions are considered for both the azimuthal shear and the radial loading. A simple deduction of the solution for one type of boundary conditions from the solution for the other type is discussed. An analysis of an axisymmetric problem in which the radial and circumferential displacements both occur simultaneously is presented by considering a thin circular disk mounted to a rigid shaft which rotates nonuniformly around its axis of symmetry.

Keywords: Airy stress function, angular velocity and acceleration, azimuthal shear, boundary conditions, Cauchy equations, compatibility conditions, disk, displacement, duality, elasticity, equilibrium, Lamé problem, Navier equations, plastic threshold, pressure, radial loading, rotation, stress

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1. INTRODUCTION

The Navier equations of equilibrium for plane strain isotropic elasticity in which the non-vanishing radial and circumferential displacements u_r and u_θ depend only on the radial coordinate r follow from general three-dimensional equations listed, e.g., in [1]-[8], and are given by

$$(\lambda + 2\mu)\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) = 0, \quad \mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2}\right) = 0, \tag{1}$$

where λ and μ are the Lamé elastic constants, and $\nabla^2 = \mathrm{d}^2/\mathrm{d}r^2 + r^{-1}\mathrm{d}/\mathrm{d}r$ is the Laplacian operator. Thus, both equations are the second-order ordinary differential equations of the Cauchy-Euler equidimensional type,

$$\frac{\mathrm{d}^2\mathrm{u_r}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{du_r}}{\mathrm{d}r} - \frac{u_r}{r^2} = 0\,, \quad \frac{\mathrm{d}^2\mathrm{u}_\theta}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{du}_\theta}{\mathrm{d}r} - \frac{u_\theta}{r^2} = 0\,. \tag{2}$$

They can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{du_r}}{\mathrm{d}r} + \frac{u_r}{r}\right) = 0\,, \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{du}_\theta}{\mathrm{d}r} + \frac{u_\theta}{r}\right) = 0\,, \tag{3}$$

with their first integrals

$$\frac{\mathrm{du_r}}{\mathrm{d}r} + \frac{u_r}{r} = 2c_1\,, \quad \frac{\mathrm{du}_\theta}{\mathrm{d}r} + \frac{u_\theta}{r} = 2k_1\,, \tag{4}$$

where $2c_1$ and $2k_1$ are the integration constants. Upon second integration, the general expressions for the displacements are

$$u_r = c_1 r + \frac{c_2}{r}\,, \quad u_\theta = k_1 r + \frac{k_2}{r}\,, \tag{5}$$

with c_2 and k_2 representing the second pair of integration constants. The equality $\mathrm{du}/\mathrm{d}r + u/r \equiv (1/r)\mathrm{d}(ru)/\mathrm{d}r$, for both u_r and u_θ, enables in (4) their direct integration.

The duality of the differential equations (2) and the expressions for the displacements (5) of these two basic but fundamentally important isotropic elasticity problems does not hold for inhomogeneous or anisotropic elastic materials. For example, for anisotropic material which is at any point of a cylinder locally orthotropic, with the principal axes of orthotropy in the (r, θ, z) directions [9, 10], the governing differential equation for circumferential displacement in the azimuthal shear problem is still given by the second equation in (2) or (4), and thus u_θ is still given by the second expression in (5), because the azimuthal shear is a statically determinate problem, and the only change in the analysis is to replace μ with $\mu_{r\theta}$. On the other hand, for the Lamé problem the governing differential equation for the radial displacement becomes

$$\frac{\mathrm{d}^2\mathrm{u_r}}{\mathrm{d}r^2} + \frac{1}{r}\,\frac{\mathrm{du_r}}{\mathrm{d}r} - n^2\,\frac{u_r}{r^2} = 0\,, \quad n^2 = \frac{E_\theta}{E_r}\,\frac{1-\nu_{rz}\nu_{zr}}{1-\nu_{\theta z}\nu_{z\theta}}\,, \tag{6}$$

with the obvious notation for different elastic moduli and Poisson's ratios of orthotropic material. Consequently, upon integration, the expression for the radial displacement is

$$u_r = c_1 r^n + \frac{c_2}{r^n}\,, \tag{7}$$

which replaces the first expression in (5) and reduces to it in the case of isotropic material ($n = 1$).

2. FIRST-ORDER DIFFERENTIAL EQUATION FOR RADIAL DISPLACEMENT

The Cauchy equilibrium equation in terms of radial and circumferential stresses for the Lamé problem of radial loading of a hollow cylinder is

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0\,. \tag{8}$$

Assuming isotropy and plane strain conditions (the strain $\epsilon_{zz} = 0$), the stresses are related to strains by generalized Hooke's law

$$\sigma_{rr} = 2\mu\epsilon_{rr} + \lambda(\epsilon_{rr} + \epsilon_{\theta\theta})\,, \quad \sigma_{\theta\theta} = 2\mu\epsilon_{\theta\theta} + \lambda(\epsilon_{rr} + \epsilon_{\theta\theta})\,. \tag{9}$$

The strains can be expressed in terms of the radial displacement by

$$\epsilon_{rr} = \frac{\mathrm{d}u_r}{\mathrm{d}r}\,, \quad \epsilon_{\theta\theta} = \frac{u_r}{r}\,, \tag{10}$$

with the corresponding strain compatibility condition

$$r\,\frac{\mathrm{d}\epsilon_{\theta\theta}}{\mathrm{d}r} = \epsilon_{rr} - \epsilon_{\theta\theta}\,. \tag{11}$$

To express the compatibility condition (11) in terms of stresses, we first use the plane strain Hooke's law to conveniently write

$$\epsilon_{\theta\theta} = \frac{1}{2\mu}\left[(1-\nu)(\sigma_{rr} + \sigma_{\theta\theta}) - \sigma_{rr}\right], \quad \epsilon_{rr} - \epsilon_{\theta\theta} = \frac{1}{2\mu}\,(\sigma_{rr} - \sigma_{\theta\theta})\,, \tag{12}$$

where ν denotes the Poisson ratio of isotropic elastic material. By substituting (12) into (11), and by using the equilibrium equation (8), the compatibility condition (11) is expressed in terms of stresses as

$$\frac{\mathrm{d}}{\mathrm{d}r}\,(\sigma_{rr} + \sigma_{\theta\theta}) = 0\,. \tag{13}$$

Thus,

$$\sigma_{rr} + \sigma_{\theta\theta} = 2m_1\,, \quad m_1 = \text{const.}\,, \tag{14}$$

i.e., the hydrostatic part of stress is uniform throughout the cylinder and qual to $2(1+\nu)m_1/3$, recalling that for the plane strain $\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$. See also [11, 12].

Equation (14) can be used to re-derive the first-order differential equation for the radial displacement u_r. Indeed, by substituting (9) and (10) into (14), it follows that

$$\frac{\mathrm{d}u_r}{\mathrm{d}r} + \frac{u_r}{r} = 2c_1\,, \quad 2c_1 = \frac{m_1}{\lambda+\mu} = \frac{(1-2\nu)m_1}{\mu}\,, \tag{15}$$

reproducing the first equation in (4). Consequently, upon integration, the radial displacement becomes

$$u_r = c_1 r + \frac{c_2}{r}\,, \tag{16}$$

which is the first expression in (5). The strains follow from (10) and the stresses from (9). They are given by

$$\epsilon_{rr} = c_1 - \frac{c_2}{r^2}\,, \quad \epsilon_{\theta\theta} = c_1 + \frac{c_2}{r^2}\,, \tag{17}$$

$$\sigma_{rr} = 2\mu\left(\frac{c_1}{1-2\nu} - \frac{c_2}{r^2}\right), \quad \sigma_{\theta\theta} = 2\mu\left(\frac{c_1}{1-2\nu} + \frac{c_2}{r^2}\right). \tag{18}$$

In the case of plane stress ($\sigma_{zz} = 0$), the denominator $1-2\nu$ is replaced with $(1-\nu)/(1+\nu)$. In this case, by Hooke's law, the out-of-plane longitudinal strain $\epsilon_{zz} = \partial u_z/\partial z$ and the displacement are

$$\epsilon_{zz} = -\frac{\nu}{1-\nu}\,(\epsilon_{rr} + \epsilon_{\theta\theta}) = -\frac{2\nu}{1-\nu}\,c_1\,, \quad u_z = -\frac{2\nu}{1-\nu}\,c_1 z\,, \tag{19}$$

with $u_z(z=0) = 0$ by symmetry around the mid-plane $z = 0$.

2.1. Stress function

In the stress formulation of the boundary-value problem, it is common to introduce the stress function $\varphi = \varphi(r)$, such that (e.g., [4], p. 343)

$$\sigma_{rr} = \frac{\varphi}{r}\,, \quad \sigma_{\theta\theta} = \frac{\mathrm{d}\varphi}{\mathrm{d}r}\,. \tag{20}$$

The Cauchy equilibrium equation (8) is then identically satisfied, as can be recognized most directly by rewriting (8) as $\mathrm{d}(r\sigma_{rr})/\mathrm{d}r = \sigma_{\theta\theta}$. The compatibility condition (14) defines the differential equation for φ,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} + \frac{\varphi}{r} = 2m_1\,, \quad m_1 = \text{const.} \tag{21}$$

Its solution is

$$\varphi = m_1 r + \frac{m_2}{r}\,, \tag{22}$$

where m_1 and m_2 are constants. Thus, from (20) and (22), the stresses become

$$\sigma_{rr} = m_1 + \frac{m_2}{r^2}\,, \quad \sigma_{\theta\theta} = m_1 - \frac{m_2}{r^2}\,. \tag{23}$$

Because by the plane strain Hooke's law,

$$\epsilon_{\theta\theta} = \frac{1}{2\mu}\left[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{\theta\theta})\right], \tag{24}$$

from $u_r = r\epsilon_{\theta\theta}$ it follows that the radial displacement is

$$u_r = \frac{1}{2\mu}\left[(1-2\nu)m_1 - \frac{m_2}{r}\right]. \tag{25}$$

By comparing (25) with (16), the constants m_1 and m_2 are related to c_1 and c_2 by $m_1 = 2\mu c_1/(1-2\nu)$ and $m_2 = -2\mu c_2$.

Remark 1: An alternative stress function $\psi = \psi(r)$ can be introduced by requiring that

$$\sigma_{rr} = \frac{\mathrm{d}\psi}{\mathrm{d}r}\,, \quad \sigma_{\theta\theta} = \frac{\psi}{r}\,, \tag{26}$$

because then both the Cauchy equilibrium equation (8) and the compatibility condition (13) reduce to the second-order equidimensional equation

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r} - \frac{\psi}{r^2} = 0\,, \tag{27}$$

whose solution is

$$\psi = m_1 r - \frac{m_2}{r}\,, \tag{28}$$

in duality with (22).

Remark 2: If the Airy stress function $\Phi = \Phi(r)$ is introduced such that, e.g., [1]-[8],

$$\sigma_{rr} = \frac{1}{r}\frac{\mathrm{d}\Phi}{\mathrm{d}r}\,, \quad \sigma_{\theta\theta} = \frac{\mathrm{d}^2\Phi}{\mathrm{d}r^2}\,, \tag{29}$$

the Cauchy equilibrium equation (8) is identically satisfied, while the compatibility condition (13) becomes a third-order differential equation $\mathrm{d}(\nabla^2\Phi)/\mathrm{d}r = 0$, i.e.,

$$\frac{\mathrm{d}^3\Phi}{\mathrm{d}r^3} + \frac{1}{r}\frac{\mathrm{d}^2\Phi}{\mathrm{d}r^2} - \frac{1}{r^2}\frac{\mathrm{d}\Phi}{\mathrm{d}r} = 0\,. \tag{30}$$

Its general solution, apart from a constant term, is

$$\Phi = \frac{1}{2} m_1 r^2 + m_2 \ln r \,. \tag{31}$$

In retrospect, by comparing (29) with (20), or (31) with (22), the stress functions φ and Φ are obviously related by $\varphi = \mathrm{d}\Phi/\mathrm{d}r$.

Remark 3: If one would require Φ to be a biharmonic function ($\nabla^4\Phi = 0$), as in non-axisymmetric two-dimensional elasticity, i.e.,

$$\frac{\mathrm{d}^4\Phi}{\mathrm{d}r^4} + \frac{2}{r}\frac{\mathrm{d}^3\Phi}{\mathrm{d}r^3} - \frac{1}{r^2}\frac{\mathrm{d}^2\Phi}{\mathrm{d}r^2} + \frac{1}{r^3}\frac{\mathrm{d}\Phi}{\mathrm{d}r} = 0 \,, \tag{32}$$

then its general solution would be, apart from a constant term,

$$\Phi = \frac{1}{2} m_1 r^2 + m_2 \ln r + m_3 r^2 \ln r \,. \tag{33}$$

The sum of normal stresses corresponding to (33) is

$$\sigma_{rr} + \sigma_{\theta\theta} = 2m_1 + 4m_3(1 + \ln r) \,, \tag{34}$$

which is not constant, as required by (14), unless $m_3 = 0$. Thus, the biharmonic term $r^2 \ln r$ does not give an admissible stress field for axisymmetric stress field with $u_r = u_r(r)$ and $u_\theta = 0$, because the corresponding strains do not satisfy the compatibility condition (11), i.e., $\Phi = r^2 \ln r$ does not satisfy the third-order differential equation for Φ, given by (30). If the stress distribution is axisymmetric, but the displacements are not, as occurs in pure bending of circular rings, or the presence of initial stresses in a closed ring produced by a disclination-type cut and weld operation (e.g., [1], p. 78-80; [2], p. 244-245; [5], p. 210-212) the constant $m_3 \neq 0$ and the compatibility condition is indeed the requirement that $\sigma_{rr} + \sigma_{\theta\theta}$ is a harmonic function, $\nabla^2(\sigma_{rr} + \sigma_{\theta\theta}) = 0$, leading to biharmonic equation for the Airy stress function ($\nabla^4\Phi = 0$), and thus its contribution proportional to $r^2 \ln r$. The use of Love's and Boussinesq's displacement potentials to solve the Lamé problem has been discussed in [13]. The Lamé strain potential $\chi = \chi(r, z)$, introduced such that $\nabla^2\chi = \mathrm{const.}$ and $\mathbf{u} = \boldsymbol{\nabla}\chi$, is $\chi = c_1 r^2/2 + c_2 \ln r + c_3 z^2/2$, which reproduces (16) for $u_r = \partial\chi/\partial r$, and gives $u_z = \partial\chi/\partial z = c_3 z$, with $c_3 = 0$ for plane strain, and c_3 related to c_1 and ν as in (19) for plane stress.

3. FIRST-ORDER DIFFERENTIAL EQUATION FOR CIRCUMFERENTIAL DISPLACEMENT

In contrast to a statically indeterminate Lamé problem, the azimuthal shear of a hollow circular cylinder is a statically determinate problem, because the Cauchy equilibrium equation,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\sigma_{r\theta}\right) = 0\,, \tag{35}$$

involves only the shear stress $\sigma_{r\theta}$, which is, by integration,

$$\sigma_{r\theta} = \frac{k}{r^2}\,, \quad k = \text{const.} \tag{36}$$

The constant k can be related to the applied torque T which produces the azimuthal shear by the moment equilibrium condition

$$T = 2\pi r^2\sigma_{r\theta} = 2\pi k \quad \Rightarrow \quad k = \frac{T}{2\pi}\,. \tag{37}$$

Furthermore, from Hooke's law the shear stress is related to shear strain by

$$\sigma_{r\theta} = 2\mu\epsilon_{r\theta}\,, \quad \epsilon_{r\theta} = \frac{1}{2}\left(\frac{\mathrm{d}u_\theta}{\mathrm{d}r} - \frac{u_\theta}{r}\right)\,. \tag{38}$$

Consequently, by equating (36) and (38), the first-order differential equation for the displacement u_θ becomes

$$\frac{\mathrm{d}u_\theta}{\mathrm{d}r} - \frac{u_\theta}{r} = \frac{k}{\mu r^2}\,. \tag{39}$$

Upon integration, this gives

$$u_\theta = k_1 r + \frac{k_2}{r}\,, \quad k_2 = -\frac{k}{2\mu}\,, \tag{40}$$

confirming the second expression in (5).

It is noted that that the displacement u_θ satisfies both the first-order differential equation in (4) and the first-order differential equation (39). As a consequence, u_θ also satisfies the second-order differential equation $\mathrm{d}^2u_\theta/\mathrm{d}r^2 = 2k_2/r^3$. The equivalency of the second equation in (4) and (39) follows by rewriting (39) as

$$\frac{\mathrm{d}u_\theta}{\mathrm{d}r} + \frac{u_\theta}{r} = \frac{k}{\mu r^2} + 2\,\frac{u_\theta}{r}\,. \tag{41}$$

Both sides of (41) must be equal to $2k_1$ in order that both so-obtained equations have the same solution $u_\theta = k_1 r + k_2/r$, with $k_2 = -k/2\mu$. We also note that the Airy stress function Φ for azimuthal shear, introduced such that $\sigma_{r\theta} = r^{-2}\mathrm{d}\Phi/\mathrm{d}\theta$, is $\Phi = k\theta$, because then $\sigma_{r\theta}$ is independent of θ.

4. AZIMUTHAL SHEAR OF A HOLLOW CYLINDER

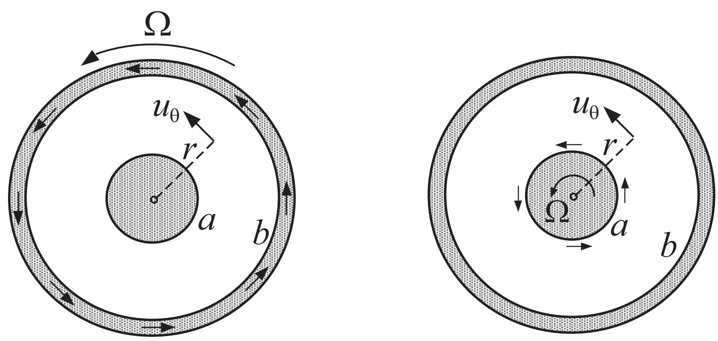
Figure 1a shows a hollow circular cylinder whose inner boundary $r = a$ is fixed, while its outer boundary $r = b$ is bonded to a rigid casing subjected to a torque T (per unit length of the cylinder) which gives rise to counter-clockwise rotation of the casing by a small angle Ω. Such loading of a hollow cylinder is referred to as azimuthal (or circular) shear (shearing), frequently studied in the past in the case of large elastic deformations of rubber-like materials, e.g., [14]-[18], where the interest was, *inter alia*, in the build-up of normal stresses accompanying the shear stress. For infinitesimal elastic strains, the normal stress effect is absent and the analysis becomes elementary. The only displacement component in cylindrical coordinates is the circumferential displacement $u_\theta = u_\theta(r)$, which is given by (40). The constants k_1 and k_2 are determined from the boundary conditions $u_\theta(a) = 0$ and $u_\theta(b) = b\Omega$, i.e.,

$$k_1 a^2 + k_2 = 0\,, \quad k_1 b^2 + k_2 = b^2\Omega\,, \tag{42}$$

which gives

$$k_1 = \frac{b^2\Omega}{b^2 - a^2}\,, \quad k_2 = -a^2 k_1\,. \tag{43}$$

Consequently, the circumferential displacement becomes



(a) (b)

Figure 1: Azimuthal shear of a hollow circular cylinder in which (a) the outer bonded casing is given a rotation Ω while the inner bonded cylinder is kept fixed, and (b) the inner cylinder is given a rotation Ω while the outer casing is fixed. The inner and outer radii of the hollow cylinder are a and b, and $u_\theta = u_\theta(r)$ is the circumferential displacement.

$$u_\theta = \frac{b^2\Omega}{b^2 - a^2}\left(r - \frac{a^2}{r}\right). \tag{44}$$

The shear stress is $\sigma_{r\theta} = 2\mu\epsilon_{r\theta}$, where, from (38) and (40), the shear strain is $\epsilon_{r\theta} = -k_2/r^2$. Thus,

$$\sigma_{r\theta} = 2\mu\,\frac{b^2\Omega}{b^2 - a^2}\,\frac{a^2}{r^2}\,. \tag{45}$$

The stresses at the inner and outer boundaries are

$$\sigma_{r\theta}(a) = 2\mu\,\frac{b^2\Omega}{b^2 - a^2}\,, \quad \sigma_{r\theta}(b) = 2\mu\,\frac{a^2\Omega}{b^2 - a^2}\,. \tag{46}$$

The corresponding torque at any radius $a \leq r \leq b$ is

$$T = 2\pi r^2 \sigma_{r\theta} = 4\pi\mu\,\frac{a^2 b^2\Omega}{b^2 - a^2}\,. \tag{47}$$

If, instead of the displacement $b\Omega$, the shear stress $\tau_b = \sigma_{r\theta}(b)$ is prescribed at the boundary $r = b$, the expressions for $u_\theta(r)$ and $\sigma_{r\theta}(r)$ follow immediately, because from the second expression in (46),

$$2\mu\,\frac{a^2\Omega}{b^2 - a^2} = \tau_b\,. \tag{48}$$

By substituting (48) into (44) and (45), it follows that

$$u_\theta = \frac{\tau_a}{2\mu}\left(r - \frac{a^2}{r}\right), \quad \sigma_{r\theta} = \tau_a\,\frac{a^2}{r^2}\,, \tag{49}$$

where $\tau_a = (b^2/a^2)\tau_b$ is the reactive (clockwise) shear stress at the inner boundary, $\tau_a = \sigma_{r\theta}(a)$. The shear stress τ_a becomes large as a becomes small relative to b, i.e., it is $(b/a)^2$ greater than the applied stress τ_b.

4.1. Deduction of the solution for the second type of boundary conditions

If the boundary conditions are as shown in Fig. 1b, i.e., if the outer boundary of the cylinder is fixed, while the inner boundary rotates in a counter-clockwise direction by Ω, the solution can be readily derived by following the same steps as described above. However, because of circular geometry and the duality of boundary conditions in Fig. 1a and 1b, the solution to the problem in Fig. 1b can be recognized immediately from the solution to the problem in Fig. 1a by interchanging a and b in the latter expressions. Thus, from (43) and (44) it follows that

$$k_1 = \frac{a^2\Omega}{a^2 - b^2}\,, \quad k_2 = -b^2 k_1\,, \tag{50}$$

$$u_\theta = \frac{a^2\Omega}{a^2 - b^2}\left(r - \frac{b^2}{r}\right). \tag{51}$$

The shear stresses follow from from (45) and (46), and are given by

$$\sigma_{r\theta} = 2\mu\,\frac{a^2\Omega}{a^2 - b^2}\,\frac{b^2}{r^2}\,, \tag{52}$$

$$\sigma_{r\theta}(b) = 2\mu\,\frac{a^2\Omega}{a^2 - b^2}\,, \quad \sigma_{r\theta}(a) = 2\mu\,\frac{b^2\Omega}{a^2 - b^2}\,. \tag{53}$$

While $u_\theta(r)$ in (51) is positive, the shear stress $\sigma_{r\theta}$ in (52) is negative, i.e., clockwise on the outer surface $r = \mathrm{const}$. The counter-clockwise torque applied to the bonded rigid cylinder of radius a is $T = 2\pi r^2|\sigma_{r\theta}|$, giving the same expression as in (47).

If, instead of the displacement $a\Omega$, the counter-clockwise shear stress $\tau_a = -\sigma_{r\theta}(a)$ is prescribed at the inner boundary $r = a$, the expressions for $u_\theta(r)$ and $\sigma_{r\theta}(r)$ follow from (49) by replacing a with b, and τ_a with $-\tau_b$. This gives

$$u_\theta = -\frac{\tau_b}{2\mu}\left(r - \frac{b^2}{r}\right), \quad \sigma_{r\theta} = -\tau_b\,\frac{b^2}{r^2}\,, \tag{54}$$

where $\tau_b = (a^2/b^2)\tau_a$ is the magnitude of the reactive shear stress at the outer boundary, $\tau_b = -\sigma_{r\theta}(b)$.

4.2. Relative rotation

If a bonded rigid cylinder of radius a rotates clockwise (rather than counter-clockwise) by Ω, the angle Ω in expressions (50)–(53) is replaced with $-\Omega$. In this case, $\sigma_{r\theta}$ is positive, while u_θ becomes negative and differs from u_θ of the problem in Fig. 1a by the rotation-induced rigid-body displacement Ωr. Indeed, denoting u_θ from (44) by u_θ^I, and $-u_\theta$ from (51) by u_θ^{II}, we have

$$u_\theta^I = \frac{b^2\Omega}{b^2 - a^2}\left(r - \frac{a^2}{r}\right), \quad u_\theta^{II} = -\frac{a^2\Omega}{a^2 - b^2}\left(r - \frac{b^2}{r}\right), \tag{55}$$

and one can readily verify that $u_\theta^{II} = u_\theta^I - \Omega r$.

One can also determine the rotations of the rigid casings at $r = a$ and $r = b$ required in order that $u_\theta(r_0) = 0$, for an arbitrary $a \le r_0 \le b$. Denoting by Ω_a the clockwise rotation of the inner casing, and by Ω_b the counter-clockwise rotation of the outer casing, they can be related to the previously introduced angle of rotation Ω by requiring that $\Omega_a + \Omega_b = \Omega$ (relative rotation of two casings), and by using the displacement expression

$$u_\theta = \frac{\tau_0}{2\mu}\left(r - \frac{r_0^2}{r}\right) = \frac{a^2b^2\Omega}{b^2 - a^2}\,\frac{1}{r_0^2}\left(r - \frac{r_0^2}{r}\right), \tag{56}$$

where $\tau_0 r_0^2 = \tau_a a^2 = \tau_b b^2$. The shear stress is $\sigma_{r\theta} = \tau_0 r_0^2/r^2$, where $\tau_0 = \sigma_{r\theta}(r_0)$. Imposing the conditions

$$a\Omega_a = -u_\theta(a) = \frac{a^2 b^2 \Omega}{b^2 - a^2}\,\frac{r_0^2 - a^2}{a r_0^2}\,, \quad b\Omega_b = u_\theta(b) = \frac{a^2 b^2 \Omega}{b^2 - a^2}\,\frac{b^2 - r_0^2}{b r_0^2}\,, \tag{57}$$

it follows that

$$\Omega_a = \frac{b^2\Omega}{r_0^2}\,\frac{r_0^2 - a^2}{b^2 - a^2}\,, \quad \Omega_b = \frac{a^2\Omega}{r_0^2}\,\frac{b^2 - r_0^2}{b^2 - a^2}\,. \tag{58}$$

5. LAMÉ PROBLEM OF A HOLLOW CYLINDER

Figure 2a shows a hollow circular cylinder whose inner boundary $r = a$ is fixed and its outer boundary $r = b$ is given a small outward radial displacement U_b. Plane strain is assumed and the elastic material is compressible ($\nu < 1/2$). The only displacement component in the cylinder is the radial displacement $u_r = u_r(r)$, which is given by (16). The constants c_1 and c_2 are determined from the boundary conditions $u_r(a) = 0$ and $u_r(b) = U_b$, i.e.,

$$c_1 a^2 + c_2 = 0\,, \quad c_1 b^2 + c_2 = bU_b\,, \tag{59}$$

which gives

$$c_1 = \frac{bU_b}{b^2 - a^2}\,, \quad c_2 = -a^2 c_1\,. \tag{60}$$

Consequently, the radial displacement becomes

$$u_r = \frac{bU_b}{b^2 - a^2}\left(r - \frac{a^2}{r}\right). \tag{61}$$

This is an analogous expression to (44) for the circumferential displacement u_θ of azimuthal shear problem, with U_b replacing $b\Omega$.

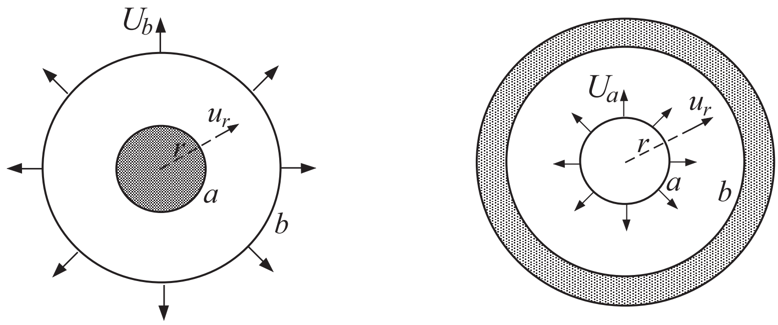
The radial and circumferential stresses are obtained by substituting c_1 and c_2 from (60) into (18),

$$\sigma_{rr} = 2\mu\,\frac{bU_b}{b^2 - a^2}\left(\frac{1}{1 - 2\nu} + \frac{a^2}{r^2}\right), \quad \sigma_{\theta\theta} = 2\mu\,\frac{bU_b}{b^2 - a^2}\left(\frac{1}{1 - 2\nu} - \frac{a^2}{r^2}\right). \tag{62}$$

The maximum shear stress at an arbitrary $a \leq r \leq b$ is

$$\tau_{\max}(r) = \frac{1}{2}\,(\sigma_{rr} - \sigma_{\theta\theta}) = 2\mu\,\frac{bU_b}{b^2 - a^2}\,\frac{a^2}{r^2}\,. \tag{63}$$

This is an analogous expression to (45) for the shear stress of azimuthal shear problem, with U_b replacing $b\Omega$. Thus, in the case of the Tresca yield criterion [3, 8], the



(a) (b)

Figure 2: Radial loading of a hollow circular cylinder in which (a) the outer boundary $r = b$ is given an outward radial displacement U_b while the inner bonded cylinder is kept fixed, and (b) the inner boundary $r = a$ is given an outward radial displacement U_a while the outer casing is fixed. The radial displacement at an arbitrary r is denoted by $u_r = u_r(r)$.

onset of plastic deformation in two problems occurs at the critical angle of rotation and critical radial displacement given by

$$\Omega^Y = \left(1 - \frac{a^2}{b^2}\right)\frac{\sigma_Y}{4\mu}\,, \quad U_b^Y = b\Omega^Y\,, \tag{64}$$

where σ_Y is the yield stress of the material in uniaxial tension.

The work done by $\sigma_{rr}(b)$ on the displacement U_b, which equals the elastic strain energy stored in the cylinder (per unit its length), is $W = (1/2)2\pi b\sigma_{rr}(b)U_b$. After using (62), this is

$$W = 2\pi\mu\,\frac{b^2U_b^2}{b^2 - a^2}\left(\frac{1}{1-2\nu} + \frac{a^2}{b^2}\right). \tag{65}$$

5.1. Deduction of the solution for the second type of boundary conditions

If the boundary conditions are as shown in Fig. 2b, i.e., if the outer boundary of the cylinder is fixed, while the inner boundary is given an outward radial displacement U_a, the solution can be readily derived by following the same steps as described above. However, because of circular geometry and the duality of boundary conditions in Fig. 2a and 2b, the solution to problem in Fig. 2b can be recognized immediately from the solution to problem in Fig. 2a by interchanging a and b in the latter expressions and by replacing U_b with U_a. Thus, from (60) and (61),

$$c_1 = \frac{aU_a}{a^2 - b^2}\,, \quad c_2 = -b^2 c_1\,, \tag{66}$$

$$u_r = \frac{aU_a}{a^2 - b^2}\left(r - \frac{b^2}{r}\right). \tag{67}$$

Expression (67) is an analogous expression to (51) for the circumferential displacement u_θ of azimuthal shear problem, with U_a replacing $a\Omega$.

The normal stresses follow directly from (62) by interchanging a and b, and by replacing U_b with U_a. This gives

$$\sigma_{rr} = 2\mu\,\frac{aU_a}{a^2 - b^2}\left(\frac{1}{1-2\nu} + \frac{b^2}{r^2}\right), \quad \sigma_{\theta\theta} = 2\mu\,\frac{aU_a}{a^2 - b^2}\left(\frac{1}{1-2\nu} - \frac{b^2}{r^2}\right). \tag{68}$$

The maximum shear stress at any $a \leq r \leq b$ is

$$\tau_{\max}(r) = \frac{1}{2}\left(\sigma_{rr} - \sigma_{\theta\theta}\right) = 2\mu\,\frac{aU_a}{a^2 - b^2}\,\frac{b^2}{r^2}\,, \tag{69}$$

in duality with (63). Expression (69) is also an analogous expression to (52) for the shear stress of the azimuthal shear problem, with U_a replacing $a\Omega$. The critical angle of rotation and critical radial displacement at the onset of plastic deformation are

$$\Omega^Y = \left(1 - \frac{a^2}{b^2}\right)\frac{\sigma_Y}{4\mu}\,, \quad U_a^Y = a\Omega^Y\,, \tag{70}$$

in duality with (64).

The work done by $\sigma_{rr}(a)$ on the displacement U_a is $W = (1/2)2\pi a|\sigma_{rr}(a)|U_a$, i.e., after using (67),

$$W = 2\pi\mu\,\frac{a^2 U_a^2}{b^2 - a^2}\left(\frac{1}{1-2\nu} + \frac{b^2}{a^2}\right). \tag{71}$$

5.2. Relationships between the solutions for different boundary conditions

Denoting the radial displacement and the stress components in the problem with the inner boundary fixed and the outer boundary radially displaced by U_b as u_r^I, σ_{rr}^I, $\sigma_{\theta\theta}^I$, $\tau_{\max}^I$, and those in the problem with the outer boundary fixed and the inner boundary radially displaced by U_a as u_r^{II}, σ_{rr}^{II}, $\sigma_{\theta\theta}^{II}$, $\tau_{\max}^{II}$, the following relationships hold between the two sets of expressions

$$u_r^{II}|_{U_a=-(a/b)U_b} = u_r^I + u_r^0 \,, \tag{72}$$

$$\sigma_{rr}^{II}|_{U_a=-(a/b)U_b} = \sigma_{rr}^I - p^0 \,, \quad \sigma_{\theta\theta}^{II}|_{U_a=-(a/b)U_b} = \sigma_{\theta\theta}^I - p^0 \,, \tag{73}$$

$$\tau_{\max}^{II}|_{U_a=-(a/b)U_b} = \tau_{\max}^I \,. \tag{74}$$

In these expressions, p^0 is a uniform (hydrostatic) pressure applied to both the inner and outer boundary of a hollow cylinder, producing the inward radial displacement of magnitude U_b at the outer boundary, i.e.,

$$p^0 = \frac{2\mu}{1-2\nu}\,\frac{U_b}{b} \,, \quad u_r^0 = -U_b\,\frac{r}{b} \,. \tag{75}$$

Because the stresses in problem I and problem II with $U_a = -(a/b)U_b$ differ by a hydrostatic pressure p^0 only, the plastic yield threshold is the same in both problems, i.e., $|U_a^Y| = (a/b)U_b^Y$, in agreement with (64) and (70).

6. AXISYMMETRIC PROBLEMS WITH RADIAL AND CIRCUMFERENTIAL DISPLACEMENTS

There are axisymmetric problems in which both displacement components, $u_r(r)$ and $u_\theta(r)$, occur simultaneously. For example if a hollow circular disk is mounted on a rigid shaft which rotates clockwise with a constant angular acceleration α, then at the instant when its angular velocity is ω, the governing differential equations for the normal stresses σ_{rr} and $\sigma_{\theta\theta}$, arising from the centrifugal inertia force due to ω, and the shear stress $\sigma_{r\theta}$, arising from the circumferential force due to α, are

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = -\rho\omega^2 r \,, \quad \frac{\mathrm{d}\sigma_{r\theta}}{\mathrm{d}r} + \frac{2\sigma_{r\theta}}{r} = -\rho\alpha r \,, \tag{76}$$

where ρ is the mass density of the disk. The second equation can be integrated as it stands to obtain

$$\sigma_{r\theta} = \frac{\rho\alpha b^2}{4}\left(\frac{b^2}{r^2} - \frac{r^2}{b^2}\right) . \tag{77}$$

The imposed boundary condition is $\sigma_{r\theta}(b) = 0$. The corresponding circumferential displacement $v = u_\theta(r) - u_\theta(a)$, relative to the surface of the shaft, readily follows from Hooke's law and the strain-displacement relation (e.g., [19]) and is given by

$$v = \frac{\rho\alpha b^3}{8\mu} \left[\left(\frac{a^2}{b^2} + \frac{b^2}{a^2} \right) \frac{r}{b} - \frac{b}{r} - \frac{r^3}{b^3} \right], \tag{78}$$

with the no-slip boundary condition imposed, $v(a) = 0$. Expression (77) can also be derived without solving the differential equation, by using the integrated form of the dynamic moment equilibrium condition,

$$2\pi\rho\alpha \int_a^r \varrho^3 \, \mathrm{d}\varrho + 2\pi[r^2\sigma_{r\theta}(r) - a^2\sigma_{r\theta}(a)] = 0\,, \tag{79}$$

where ϱ is a dummy integration variable. Integrating (79) from a to $r = b$, and using $\sigma_{r\theta}(b) = 0$, gives the expression for the shear stress at $r = a$,

$$\sigma_{r\theta}(a) = \frac{\rho\alpha b^2}{4} \left(\frac{b^2}{a^2} - \frac{a^2}{b^2} \right). \tag{80}$$

If (80) is substituted back into (79), the integration from a to r gives the expression for the shear stress at an arbitrary $a \leq r \leq b$, given by (77). For an analysis of accelerating solid disk, see [20], p. 88-90.

Regarding the displacement and stresses from the angular velocity, by implementing the strain-displacement relations and the plane stress Hooke's law into the first equation in (76) yields the differential equation for u_r (e.g., [7, 9])

$$\frac{\mathrm{d}^2 \mathrm{u_r}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{du_r}}{\mathrm{d}r} - \frac{u_r}{r^2} = -\frac{1-\nu}{2\mu}\,\rho\omega^2 r\,. \tag{81}$$

Its solution is

$$u_r = c_1 r + \frac{c_2}{r} - \frac{1-\nu}{16\mu}\,\rho\omega^2 r^3\,, \tag{82}$$

generalizing (16). The corresponding strains are

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial u_r}{\partial r} = c_1 - \frac{c_2}{r^2} - \frac{3(1-\nu)}{16\mu}\,\rho\omega^2 r^2\,, \\ \epsilon_{\theta\theta} &= \frac{u_r}{r} = c_1 + \frac{c_2}{r^2} - \frac{1-\nu}{16\mu}\,\rho\omega^2 r^2\,. \end{aligned} \tag{83}$$

The normal stresses then follow by substituting (83) into the plane stress Hooke's law

$$\sigma_{rr} = \frac{2\mu}{1-\nu}\left(\epsilon_{rr} + \nu\epsilon_{\theta\theta}\right), \quad \sigma_{\theta\theta} = \frac{2\mu}{1-\nu}\left(\epsilon_{\theta\theta} + \nu\epsilon_{rr}\right), \tag{84}$$

which gives

$$
\begin{aligned}
\sigma_{rr} &= 2\mu\left(\frac{1+\nu}{1-\nu}\,c_1 - \frac{c_2}{r^2}\right) - \frac{3+\nu}{8}\,\rho\omega^2 r^2\,, \\
\sigma_{\theta\theta} &= 2\mu\left(\frac{1+\nu}{1-\nu}\,c_1 + \frac{c_2}{r^2}\right) - \frac{1+3\nu}{8}\,\rho\omega^2 r^2\,,
\end{aligned}
\tag{85}
$$

generalizing the plane stress version of (18). Finally, by imposing the boundary conditions $u_r(a) = 0$ and $\sigma_{rr}(b) = 0$, the integration constants are found to be

$$
\begin{aligned}
c_1 &= \frac{(1-\nu)\rho\omega^2}{16\mu}\,\frac{(1-\nu)a^4 + (3+\nu)b^4}{(1-\nu)a^2 + (1+\nu)b^2}\,, \\
c_2 &= \frac{(1-\nu)\rho\omega^2 a^2 b^2}{16\mu}\,\frac{(1+\nu)a^2 - (3+\nu)b^2}{(1-\nu)a^2 + (1+\nu)b^2}\,.
\end{aligned}
\tag{86}
$$

The out-of-plane longitudinal strain $\epsilon_{zz} = \partial u_z/\partial z$ follows from (84) or (85) by using Hooke's law,

$$
\epsilon_{zz} = -\frac{\nu}{E}\,(\sigma_{rr} + \sigma_{\theta\theta}) = -\frac{\nu}{1-\nu}\,(\epsilon_{rr} + \epsilon_{\theta\theta}) = -\frac{2\nu}{1-\nu}\,c_1 - \frac{\nu}{4\mu}\,\rho\omega^2 r^2\,. \tag{87}
$$

The corresponding longitudinal displacement is

$$
u_z = -\left(\frac{2\nu}{1-\nu}\,c_1 + \frac{\nu}{4\mu}\,\rho\omega^2 r^2\right) z\,, \tag{88}
$$

such that $u_z(z = 0) = 0$.

6.1. Stress-based approach

The displacement and stress expressions corresponding to ω were derived in the previous section by the displacement-based approach. Alternatively, they can be derived by the stress-based approach by introducing the stress function $\varphi = \varphi(r)$ such that, e.g. [4], p. 335,

$$\sigma_{rr} = \frac{\varphi}{r}, \quad \sigma_{\theta\theta} = \frac{\mathrm{d}\varphi}{\mathrm{d}r} + \rho\omega^2 r^2 . \tag{89}$$

The Cauchy equation of motion in (76) is then identically satisfied, while the compatibility condition $\mathrm{d}(r\epsilon_{\theta\theta})/\mathrm{d}r = \epsilon_{rr}$ becomes

$$\frac{\mathrm{d}}{\mathrm{d}r}(\sigma_{rr} + \sigma_{\theta\theta}) = -(1+\nu)\rho\omega^2 r , \tag{90}$$

generalizing (21). The substitution of (89) into (90) gives a differential equation for φ,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}r} + \frac{\varphi}{r}\right) = -(3+\nu)\rho\omega^2 r . \tag{91}$$

Its solution is

$$\varphi = m_1 r + \frac{m_2}{r} - \frac{3+\nu}{8}\,\rho\omega^2 r^3 , \tag{92}$$

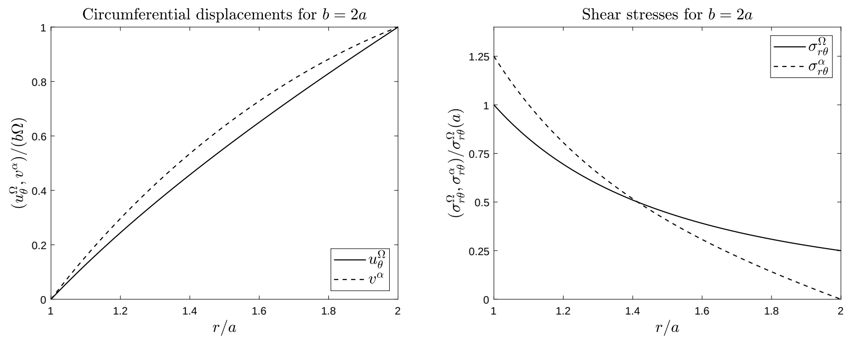
reducing to (22) when $\omega = 0$.

We note that a rotating disk is, strictly speaking, a three-dimensional elasticity problem in which σ_{rr} and $\sigma_{\theta\theta}$ also depend on the z coordinate, orthogonal to the plane of the disk. For thin disks this dependence on z is mild and for most practical purposes can be safely ignored, e.g., [1], p. 389; [8], p. 209. Also, the z-dependent stress-correction term is self-equilibrating over the thickness of the disk. The shear stress expression (77) due to the angular acceleration α is, on the other hand, exact.

6.2. Maximum shear stress in a disk due to azimuthal shear vs. spinning

It may be of design interest to evaluate and compare the displacement and shear stress distributions in a stationary disk subjected to azimuthal shear and a rotating disk subjected to angular acceleration, under the condition of equal circumferential displacements at the outer boundary of the disk relative to the inner boundary. By using (44) and (78), the condition for equal relative displacements $u_\theta^\Omega(b) = v^\alpha(b)$ gives the relationship between α and Ω,

$$\rho\alpha = \frac{8\mu a^2\Omega}{(b^2 - a^2)^2} . \tag{93}$$



(a) (b)

Figure 3: (a) The normalized displacements due to azimuthal shear Ω and angular acceleration α of a thin disk vs. r/a, in the case when α and Ω are related by the condition $u_\theta^\Omega(b) = v^\alpha(b)$. (b) The corresponding variations of the normalized shear stresses.

When (89) is substituted into (44) and (78), the normalized displacements are found to be

$$\begin{aligned} \frac{u_\theta^\Omega}{b\Omega} &= \frac{a/b}{1-(a/b)^2}\left(\frac{r}{a}-\frac{a}{r}\right), \\ \frac{v^\alpha}{b\Omega} &= \frac{(a/b)^3}{[1-(a/b)^2]^2}\left[\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)\frac{r}{a}-\left(\frac{b}{a}\right)^2\frac{a}{r}-\left(\frac{a}{b}\right)^2\left(\frac{r}{a}\right)^3\right]. \end{aligned} \tag{94}$$

They are plotted in Fig. 3a in the case $b = 2a$. The nonlinearity of $u_\theta^\Omega/b\Omega$ becomes more pronounced for smaller values a/b, and the two curves get closer to each other with the decrease of a/b. Furthermore, when (93) is substituted into (45) and (77), the normalized shear stresses become

$$\begin{aligned} \frac{\sigma_{r\theta}^\Omega}{\sigma_{r\theta}^\Omega(a)} &= \frac{a^2}{r^2}, \quad \sigma_{r\theta}^\Omega(a) = \frac{2\mu b^2\Omega}{b^2-a^2}, \\ \frac{\sigma_{r\theta}^\alpha}{\sigma_{r\theta}^\Omega(a)} &= \frac{b^2}{b^2-a^2}\,\frac{a^2}{r^2}\left(1-\frac{r^4}{b^4}\right). \end{aligned} \tag{95}$$

Their plots are shown in Fig. 3b. The ratio of the maximum shear stresses is

$$\frac{\sigma_{r\theta}^\alpha(a)}{\sigma_{r\theta}^\Omega(a)} = 1+\frac{a^2}{b^2}, \quad \frac{\sigma_{r\theta}^\Omega(b)}{\sigma_{r\theta}^\Omega(a)} = \frac{a^2}{b^2}. \tag{96}$$

Thus, for $b = 2a$, the maximum shear stress due to angular acceleration is greater by 25% than the maximum shear stress due to static azimuthal shear, under the imposed condition of equal maximum relative displacements. The curves get closer to each other with the decrease of a/b. Similar, albeit a somewhat more tedious analysis proceeds to compare the maximum shear stresses due to radial displacement U_b and angular velocity ω, when it is required that the radial displacements at $r = b$ are equal in both cases, which gives

$$\rho\omega^2 = \frac{8\mu U_b/b}{1-\nu}\,\frac{(1-\nu)a^2 + (1+\nu)b^2}{(b^2-a^2)^2}\,. \tag{97}$$

7. CONCLUSION

The governing equation for the circumferential displacement $u_\theta = u_\theta(r)$ in a hollow circular cylinder subjected to azimuthal shear is of the same type as the governing equation for the radial displacement $u_r = u_r(r)$ due to radial stretching of a hollow cylinder (Lamé problem). Both equations are the second-order equidimensional ordinary differential equations, whose solution is a linear combination of r and $1/r$ terms, where r is the radial distance from the center of the cylinder. The maximum shear stresses in both problems vary as $1/r^2$ and are given by analogous expressions, which implies that the plastic yield threshold is also defined by analogous expressions. The compatibility condition in terms of stresses for the Lamé problem is given by the condition $\mathrm{d}(\sigma_{rr} + \sigma_{\theta\theta})/\mathrm{d}r = 0$, rather than the condition that the sum of two in-plane normal stresses is a harmonic function, as in general two-dimensional elasticity. As a consequence, the Airy stress function is governed by a third-order differential equation, rather than the usual biharmonic differential equation. Two types of boundary conditions are considered for both azimuthal shear and radial loading of a hollow cylinder. A simple deduction of the solution for one type of boundary conditions from the solution for the other type is described. An analysis of axisymmetric problems in which the radial and circumferential displacements occur simultaneously is presented by considering a thin circular disk mounted to a rigid shaft which rotates around its axis with nonuniform angular velocity.

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DUALNOST AZIMUTALNOG SMICANJA I RADIJALNOG OPTEREĆENJA CILINDRIČNE CIJEVI I VEZANI PROBLEMI

S a ž e t a k

Navijeova jednačina ravnoteže za tangencijalno pomjeranje $u_\theta = u_\theta(r)$ u cilindričnoj cijevi podvrgnutoj azimutalnom smicanju istog je ekvidimenzionalnog oblika kao odgovarajuća jednačina za radijalno pomjeranje $u_r = u_r(r)$ u Lameovom problemu cilindrične cijevi pod dejstvom radijalnog opterećenja. Maksimalni smičući naponi u oba problema mijenjaju se kao $1/r^2$, gdje r označava radijalno odstojanje od centralne ose cilindra. Njihovi izrazi su analogni, što ima za posljedicu da su i izrazi za početak plastične deformacije u cilindru takođe analogni. Različite naponske funkcije za Lameov problem su uvedene i diskutovane u kontekstu nestandardnog oblika uslova kompatibilnosti, zbog kog Erijeva funkcija napona zadovoljava diferencijalnu jednačinu trećeg reda, umjesto standardnu biharmonijsku jednačinu opšteg dvodimenzionalnog problema izotropne elastičnosti u odsustvu zapreminskih sila. Dvije vrste graničnih uslova su analizirane i za azimutalno smicanje i za radijalno opterećenje cilindra. Rješenje za jednu vrstu graničnih uslova proizilazi direktno iz rješenja za drugu vrstu jednostavnim transformacijama, bez rješavanja diferencijalnih jednačina pri novim graničnim uslovima. Data je i kratka analiza aksisimetričnog problema rotacije tankog diska učvršćenog za krutu osovinu koja se obrće oko svoje ose neravnomjernom ugaonom brzinom, u kojem su istovremeno prisutne i radijalna i tangencijalna komponenta pomjeranja tačaka diska.

Ključne riječi: azimutalno smicanje, cilindar, disk, dualnost, elastičnost, Erijeva naponska funkcija, granični uslovi, Košijeve jednačine, Lameov problem, Navijeove jednačine, napon, pomjeranje, prag plastičnosti, pritisak, radijalno opterećenje, rotacija, ugaona brzina i ubrzanje, uslovi kompatibilnosti