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Vlado Lubarda[*]

ON THE KIENZLER-DUAN FORMULA FOR THE HOOP STRESS AROUND A CIRCULAR VOID

Abstract

The derivation of the Kienzler-Duan formula for the hoop stress around a circular void caused by either a remote loading or nearby internal source of stress is presented based on the Fourier series analysis without referral to the Poisson coefficient of lateral contraction, as in the original derivation by Kienzler and Duan (1987). The formula for the longitudinal stress below the free surface of a half-space due to nearby internal source of stress is also derived by means of the limiting process from the solution to the problem of an internal source of stress near an infinitely large circular void, and by an independent analysis without referral to the former problem. The use of the derived formulas in the dislocation and inclusion problems of mechanics of solids and materials science is discussed.

O KIENZLER-DUANOVOJ FORMULI ZA OBRUČNI NAPON OKO KRUŽNOG OTVORA

Sažetak

Kienzler-Duanova formula za obručni napon oko kružnog otvora usljed spoljašnjeg opterećenja ili unutrašnjeg izvora napona je izvedena na bazi Fourierove analize, bez uvođenja u analizu Poissonovog koeficijenta elastičnosti koji je korišćen u radu Kienzlera i Duana (1987). Formula za uzdužni napon ispod slobodne površine poluprostora usljed obližnjeg unutrašnjeg izvora napona je izvedena kao granični slučaj unutrašnjeg izvora napona u blizini beskonačno velikog otvora, a

[*] Prof. Dr. V. A. Lubarda, The Montenegrin Academy of Sciences and Arts, Montenegro, and University of California, San Diego, CA 92093-0448, USA.

zatim nezavisnom analizom ne pozivajući se na rješenje problema beskonačnog prostora oslabljenog kružnim otvorom. Diskutovan je značaj ovih formula za analizu dislokacionih problema i inkluzija u mehanici čvrstih tijela i nauci o materijalima.

1 Introduction

The hoop stress along the boundary of a circular void in an infinite medium under plane stress or plane strain conditions, due to remote loading or an internal source of stress, can be expressed in terms of the stress field produced by the same loading in an infinite medium without a void. This can be recognized from Green and Zerna's (1968) analysis in section 8.19, which yields an expression for the hoop stress along the boundary of the circle in terms of complex potentials, but their final expression (8.19.9) was left in a not fully disclosed form. It was in the paper by Kienzler and Duan (1987), where the explicit formula for the hoop stress was first reported in the following form

$$\sigma_\theta(a,\theta) = 2[\sigma_\theta^0(a,\theta) - \sigma_r^0(a,\theta)] + \frac{1}{2}\,[\sigma_\theta^0(0,\theta) + \sigma_r^0(0,\theta)]\,. \tag{1.1}$$

The stress field in an infinite medium without a void due to the same source of stress is denoted by the superscript ^{0}, and (r,θ) are the polar coordinates with the origin at the center of the void of radius a. In the subsequent sequence of papers, Honein and Herrmann (1988,1990) developed the so-called heterogenization procedure, according to which the solution to the problems of two or more inhomogeneities under remote or other type of loadings is expressed in terms of the solution to the corresponding homogeneous problems. They used in their analysis the complex potential approach, and also interpreted the second part of the Kienzler–Duan formula (1.1), originally left in an integral form, in terms of the first stress invariant at the center of the circle. The formula (1.1) has been applied by Kienzler and Kordisch (1990) to evaluate the interaction between a circular hole and an edge dislocation, and has been further discussed by Greenwood (1994), Golecki (1995), and Chao and Heh (1999). More recently, Lubarda (2015c) used (1.1) to evaluate the J and M integrals along the boundary of a circular void due to a nearby circular inclusion under uniform dilatational eigenstrain, and the change of the strain energy associated with the change of relative position or the size of the inclusion and void, without solving the entire boundary value problem at hand. The infinite medium solutions for the inclusions reported in Lubarda (1998) and Lubarda and Markenscoff (1999) were utilized instead.

There is an analogous formula for the circumferential shear stress along the boundary of a circular void in the case of antiplane strain, first recognized by Lin et al. (1990), which states that this circumferential shear stress is equal twice the circumferential shear stress along the corresponding circle in an infinite medium without a void, under the same loading conditions, i.e.,

$$\sigma_{z\theta}(a,\theta) = 2\sigma_{z\theta}^0(a,\theta)\,. \tag{1.2}$$

Lubarda (2015a) employed this formula to determine the configurational force between a circular void and a circular inclusion characterized by uniform eigenshear of the antiplane strain type. For noncircular voids, the ratio of the circumferential shear stress along the boundary of the void and a congruent curve in an infinite solid without a void depends on the loading, as elaborated upon in the case of an elliptical void by Lubarda (2015b).

In this paper, we shed additional light to the derivation of the formula (1.1) for the hoop stress around a circular void under plane stress or strain conditions. The derivation is based on the Fourier series analysis, but does not make a referral to Poisson's coefficient, as in equations (3a) and (3b) of the original Kienzler–Duan's (1987) derivation. In the limit as the radius of the void increases to infinity, the problem of the source of internal stress near the free surface ($y = 0$) of a half-space is deduced, in which case (1.1) simplifies to

$$\sigma_y(0,y) = 2[\sigma_y^0(0,y) - \sigma_x^0(0,y)]\,. \tag{1.3}$$

The formula (1.3) is confirmed by an independent analysis of the inclusion embedded in a half-space, without performing the aforementioned limiting process in the solution to the problem of a circular inclusion near a circular void.

2 Circumferential shear stress along the boundary of the void

Consider a circular cylindrical void of radius a in an infinitely extended isotropic elastic medium (Fig. 1). Suppose that at point C, at distance d from the center O of the void, there is a source of stress, such as an edge dislocation, a concentrated force, or an inclusion with a uniform eigenstrain. Alternatively, the source of stress around the void may be a remote loading. The inplane stress components are related to the biharmonic Airy stress function $\Phi = \Phi(r,\theta)$ by the well-known expressions

$$\sigma_r = \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2}\,, \quad \sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2}\,, \quad \sigma_{r\theta} = \frac{1}{r^2}\frac{\partial \Phi}{\partial \theta} - \frac{1}{r}\frac{\partial^2 \Phi}{\partial r \partial \theta}\,. \tag{2.1}$$

The stress field around the void in an infinite medium due to the source of stress at C can be determined by the superposition principle. First, the stress distribution is found in an infinite medium without the void due to the same source of stress at C (Fig. 1b). Denote these stresses by $\sigma_r^0(r,\theta)$, $\sigma_\theta^0(r,\theta)$, and $\sigma_{r\theta}^0(r,\theta)$. Then, the auxiliary problem is solved for the void in an infinite medium, loaded on its surface $r = a$ by the (self-equilibrating) traction $\hat{\sigma}_r(a,\theta) = -\sigma_r^0(a,\theta)$ and $\hat{\sigma}_{r\theta}(a,\theta) = -\sigma_{r\theta}^0(a,\theta)$, as sketched in Fig. 2a. The stress distribution of the original problem from Fig. 1a is the sum of the stress distributions for the problems from Figs. 1b and 2a, i.e., $\sigma_r(r,\theta) = \sigma_r^0(r,\theta) + \hat{\sigma}_r(r,\theta)$, and likewise for $\sigma_\theta(r,\theta)$ and $\sigma_{r\theta}(r,\theta)$.

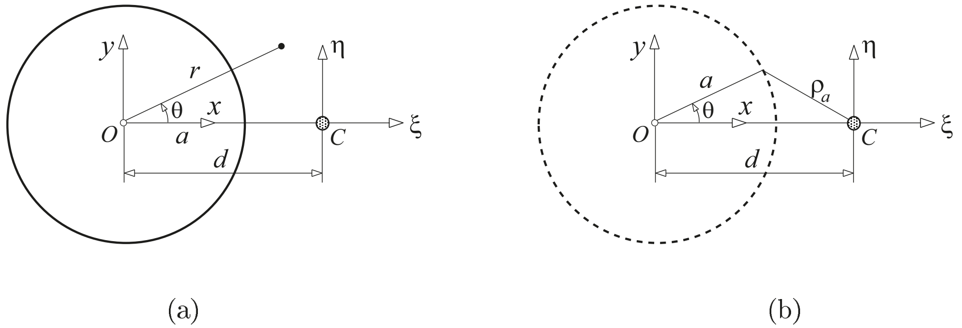


Figure 1: (a) The source of internal stress at point C at distance d from the center of the void of radius a. The polar coordinates at O are (r, θ). The cartesian coordinates at O are (x, y), while at C they are (ξ, η). (b) The source of internal stress at point C in an infinite medium without the void. The dashed-line circle of radius a coincides with the boundary of the void in a voided infinite medium from part (a).

2.1 Cylinder problem

Consider a circular cylinder of radius a loaded on its boundary by $\sigma_r^0(a,\theta)$ and $\sigma_{r\theta}^0(a,\theta)$, as shown in Fig. 2b. The state of stress within this cylinder is the same as the state of stress within the circle $r = a$ of the infinite medium problem from Fig. 1b. The stress function for the cylinder problem, chosen to give rise to unique displacements within $r \leq a$, a self-equilibrating traction at $r = a$, and no singularity at $r = 0$, is (e.g., Timoshenko and Goodier, 1970; Malvern, 1968)

$$\Phi_c = A_0 r^2 + r^3 f_1(\theta) + \sum_{n=2}^{\infty} \left[r^n f_n(\theta) + r^{n+2} B_n g_n(\theta) \right] . \tag{2.2}$$

Here,

$$f_n(\theta) = A_n \cos n\theta + C_n \sin n\theta \,, \quad g_n(\theta) = B_n \cos n\theta + D_n \sin n\theta \,, \tag{2.3}$$

which are chosen to give rise to unique displacements within $r \leq a$, a self-equilibrating traction at $r = a$, and no singularity at $r = 0$. The associated stresses, from (2.1), are

$$\sigma_r^0(r,\theta) = 2A_0 + 2r f_1(\theta) - \sum_{n=2}^{\infty} \Big[n(n-1) r^{n-2} f_n(\theta) + (n+1)(n-2) r^n g_n(\theta) \Big] , \tag{2.4}$$

$$\sigma_\theta^0(r,\theta) = 2A_0 + 6r f_1(\theta) + \sum_{n=2}^{\infty} \Big[n(n-1) r^{n-2} f_n(\theta) + (n+1)(n+2) r^n g_n(\theta) \Big] , \tag{2.5}$$

$$\sigma_{r\theta}^0(r,\theta) = -2r f_1'(\theta) - \sum_{n=2}^{\infty} \Big[(n-1) r^{n-2} f_n'(\theta) + (n+1) r^n g_n'(\theta) \Big] , \tag{2.6}$$

where the prime symbol $()'$ denotes the derivative with respect to θ. In particular, at the center $r = 0$, the stresses are

$$\sigma_r^0(0,\theta) = 2A_0 - 2f_2(\theta)\,, \quad \sigma_\theta^0(0,\theta) = 2A_0 + 2f_2(\theta)\,, \quad \sigma_{r\theta}^0(0,\theta) = -f_2'(\theta)\,. \tag{2.7}$$

The constant $2A_0$ can be interpreted as the mean inplane normal stress at that point, $2A_0 = I_1^0/2$, where $I_1^0 = \sigma_r^0(0,\theta) + \sigma_\theta^0(0,\theta)$. Along any circle or radius $r \leq a$, the average radial and hoop stresses are equal to each other, both being equal to $2A_0$, i.e.,

$$\frac{1}{2\pi}\int_0^{2\pi} \sigma_r^0(r,\theta)\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \sigma_\theta^0(r,\theta)\,\mathrm{d}\theta = 2A_0\,. \tag{2.8}$$

For the later purposes, from (2.5) it follows by inspection that

$$\sigma_\theta^0(a,\theta) - 2\sigma_r^0(a,\theta) = 2af_1(\theta) + \sum_{n=2}^{\infty}\Big[3n(n-1)a^{n-2}f_n(\theta) + (3n-2)(n+1)a^n g_n(\theta)\Big]\,. \tag{2.9}$$

2.2 Auxiliary problem

The stress field due to a self-equilibrating traction $\hat{\sigma}_r(a,\theta) = -\sigma_r^0(a,\theta)$ and $\hat{\sigma}_{r\theta}(a,\theta) = -\sigma_{r\theta}^0(a,\theta)$ of the auxiliary problem from Fig. 2a must vanish as $r \to \infty$, or give rise there to a uniform remote stress state. The corresponding stress function, yielding the unique displacements and no singularities for $r \geq a$, is

$$\hat{\Phi} = \hat{A}_0 \ln r + r^{-1}\hat{f}_1(\theta) + \sum_{n=2}^{\infty}\Big[r^{-n}\hat{f}_n(\theta) + r^{-n+2}\hat{g}_n(\theta)\Big]\,, \tag{2.10}$$

where

$$\hat{f}_n(\theta) = \hat{A}_n \cos n\theta + \hat{C}_n \sin n\theta\,, \quad \hat{g}_n(\theta) = \hat{B}_n \cos n\theta + \hat{D}_n \sin n\theta\,. \tag{2.11}$$

The associated stresses, again from (2.1), are

$$\hat{\sigma}_r(r,\theta) = \frac{\hat{A}_0}{r^2} - \frac{2}{r^3}\,\hat{f}_1(\theta) - \sum_{n=2}^{\infty}\Big[n(n+1)r^{-n-2}\hat{f}_n(\theta) + (n-1)(n+2)r^{-n}\hat{g}_n(\theta)\Big]\,, \tag{2.12}$$

$$\hat{\sigma}_\theta(r,\theta) = -\frac{\hat{A}_0}{r^2} + \frac{2}{r^3}\,\hat{f}_1(\theta) + \sum_{n=2}^{\infty}\Big[n(n+1)r^{-n-2}\hat{f}_n(\theta) + (n-1)(n-2)r^{-n}\hat{g}_n(\theta)\Big]\,, \tag{2.13}$$

$$\hat{\sigma}_{r\theta}(r,\theta) = \frac{2}{r^3}\,\hat{f}_1'(\theta) + \sum_{n=2}^{\infty}\Big[(n+1)r^{-n-2}\hat{f}_n'(\theta) + (n-1)r^{-n}\hat{g}_n'(\theta)\Big]\,. \tag{2.14}$$

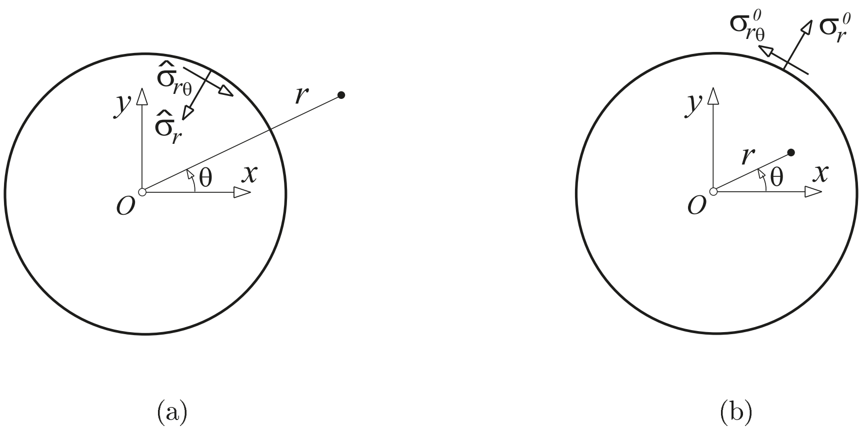


Figure 2: (a) The auxiliary problem in which the void in an infinite medium is loaded over its boundary $r = a$ with a self-equilibrating traction $\hat{\sigma}_r(a, \theta) = -\sigma^0_r(a, \theta)$ and $\hat{\sigma}_{r\theta}(a, \theta) = -\sigma^0_{r\theta}(a, \theta)$, where $\sigma^0_r(a, \theta)$ and $\sigma^0_{r\theta}(a, \theta)$ are the radial and shear stress components along the circle $r = a$ in an unvoided infinite medium. (b) A solid circular cylinder loaded over its boundary $r = a$ by the shear stress $\sigma^0_r(a, \theta)$ and $\sigma^0_{r\theta}(a, \theta)$.

2.3 Boundary conditions

The boundary conditions at $r = a$ are $\hat{\sigma}_r(a, \theta) = -\sigma^0_r(a, \theta)$ and $\hat{\sigma}_{r\theta}(a, \theta) = -\sigma^0_{r\theta}(a, \theta)$. By using the stress expressions from the previous sections, the first of the boundary conditions requires that

$$\frac{\hat{A}_0}{a^2} - \frac{2}{a^3}\,\hat{f}_1(\theta) = -2A_0 - 2af_1(\theta), \tag{2.15}$$

and

$$\begin{aligned} &-\sum_{n=2}^{\infty}\left[n(n+1)a^{-n-2}\hat{f}_n(\theta) + (n-1)(n+2)a^{-n}\hat{g}_n(\theta)\right] \\ &= \sum_{n=2}^{\infty}\left[n(n-1)a^{n-2}f_n(\theta) + (n+1)(n-2)a^n g_n(\theta)\right]. \end{aligned} \tag{2.16}$$

Similarly, the second boundary condition requires that

$$\frac{2}{a^3}\,\hat{f}_1'(\theta) = 2af_1'(\theta), \tag{2.17}$$

and

$$\sum_{n=2}^{\infty}\left[(n+1)a^{-n-2}\hat{f}_n'(\theta) + (n-1)a^{-n}\hat{g}_n'(\theta)\right] = \sum_{n=2}^{\infty}\left[(n-1)a^{n-2}f_n'(\theta) + (n+1)a^n g_n'(\theta)\right]. \tag{2.18}$$

Consequently,

$$\hat{A}_0 = -2a^2 A_0\,, \quad \hat{A}_1 = a^4 A_1\,, \quad \hat{C}_1 = a^4 C_1\,, \tag{2.19}$$

and

$$\begin{aligned}-n(n+1)a^{-n-2}\hat{A}_n - (n-1)(n+2)a^{-n}\hat{B}_n &= n(n-1)a^{n-2}A_n + (n+1)(n-2)a^n B_n\,,\\ n(n+1)a^{-n-2}\hat{A}_n + n(n-1)a^{-n}\hat{B}_n &= n(n-1)a^{n-2}A_n + n(n+1)a^n B_n\,.\end{aligned} \tag{2.20}$$

The analogous equations are obtained for the constants $(\hat{C}_n, \hat{D}_n)$ in terms of (C_n, D_n). Upon solving for $(\hat{A}_n, \hat{B}_n)$, one obtains

$$a^{-n-2}\hat{A}_n = (n-1)a^{n-2}A_n + na^n B_n\,, \quad a^{-n}\hat{B}_n = -na^{n-2}A_n - (n+1)a^n B_n\,, \tag{2.21}$$

and similarly for $(\hat{C}_n, \hat{D}_n)$ in terms of (C_n, D_n). When this is substituted into (2.13), the following expression is obtained for the hoop stress along the boundary of the void of the auxiliary problem,

$$\hat{\sigma}_\theta(a,\theta) = 2A_0 + 2af_1(\theta) + \sum_{n=2}^{\infty}\Big[3n(n-1)a^{n-2}f_n(\theta) + (3n-2)(n+1)a^n g_n(\theta)\Big]\,. \tag{2.22}$$

2.4 Kienzler–Duan's formula

By comparing (2.22) with (2.9), it follows that

$$\hat{\sigma}_\theta(a,\theta) = 2A_0 + \sigma^0_\theta(a,\theta) - 2\sigma^0_r(a,\theta)\,, \quad 2A_0 = \frac{1}{2}\left[\sigma^0_\theta(0,\theta) + \sigma^0_r(0,\theta)\right]. \tag{2.23}$$

Since the total hoop stress along the boundary of the void is

$$\sigma_\theta(a,\theta) = \sigma^0_\theta(a,\theta) + \hat{\sigma}_\theta(a,\theta)\,, \tag{2.24}$$

the substitution of (2.23) into (2.24) yields (1.1), i.e.,

$$\sigma_\theta(a,\theta) = 2[\sigma^0_\theta(a,\theta) - \sigma^0_r(a,\theta)] + \frac{1}{2}\left[\sigma^0_\theta(0,\theta) + \sigma^0_r(0,\theta)\right]. \tag{2.25}$$

This formula was first derived through a somewhat different procedure by Kienzler and Duan (1987). The determination of the hoop stress along the boundary of a circular void, based on the infinite medium stress field without the void, was previously described by Green and Zerna (1968), who derived the formula (8.19.9) in their section 8.19 in terms of complex potentials. Honein and Herrmann (1990) constructed the solution to the problem of a circular inhomogeneity within an infinite matrix in terms of the solution to the corresponding homogeneous problem. They also used in their analysis the complex potential approach, and interpreted the second part of the

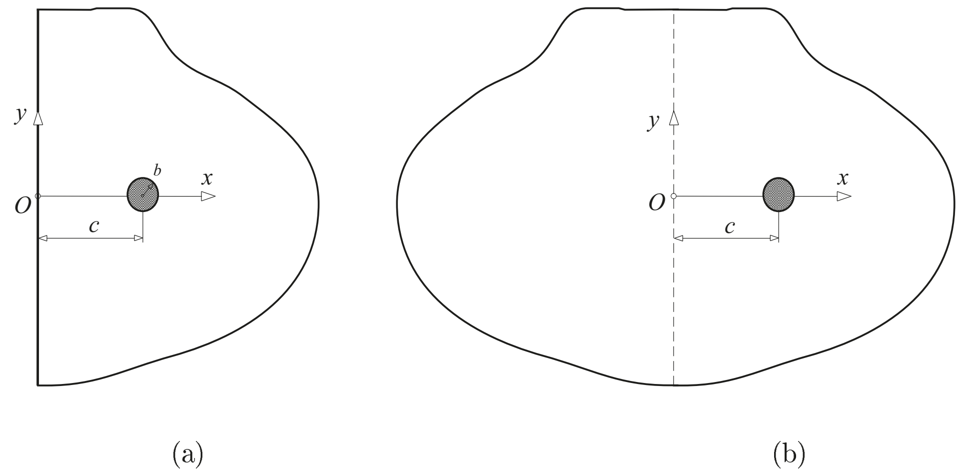


Figure 3: (a) A source of stress at distance c from the free surface $x = 0$ of a half-space. (b) An infinite medium with the same source of stress as in part (a).

Kienzler–Duan formula (2.25), originally left in an integral form, in terms of the first stress invariant at the center of the circle.

In the limit as the radius of the void increases to infinity, the problem of the source of internal stress near the free surface of a half-space is deduced. In this case, the formula (2.25) reduces to $\sigma_y(0,y) = 2[\sigma_y^0(0,y) - \sigma_x^0(0,y)]$, because the stresses in an infinite medium far away from the source of internal stress decay to zero. An independent derivation of this result is given in the following section.

3 Longitudinal stress below the free surface of a half-space

Figure 3a shows the source of stress (e.g., an inclusion, an edge dislocation, a concentrated force, a doublet of forces, or a center of dilatation) near the free surface of a half-space, at distance c from it. The objective is to determine the longitudinal stress $\sigma_y(0,y)$ along the free surface $x = 0$ by using only the solution to the problem of the same source of stress embedded within an infinite medium (Fig. 3b). If the infinite medium from Fig. 3b is imagined to be divided along $x = 0$, the two configurations (left and right portion of the infinite medium) are obtained, shown in 4. For the left portion (Fig. 4a), the longitudinal stress along $x = 0$ can be written as the sum of the contributions from the applied self-equilibrating tractions $\sigma_x(0,y)$ and $\sigma_{xy}(0,y)$ (denoted for convenience by σ_0 and τ_0), i.e.,

$$\sigma_y^0(0,y) = \sigma_y^{\sigma_0}(0,y) + \sigma_y^{\tau_0}(0,y)\,. \tag{3.1}$$

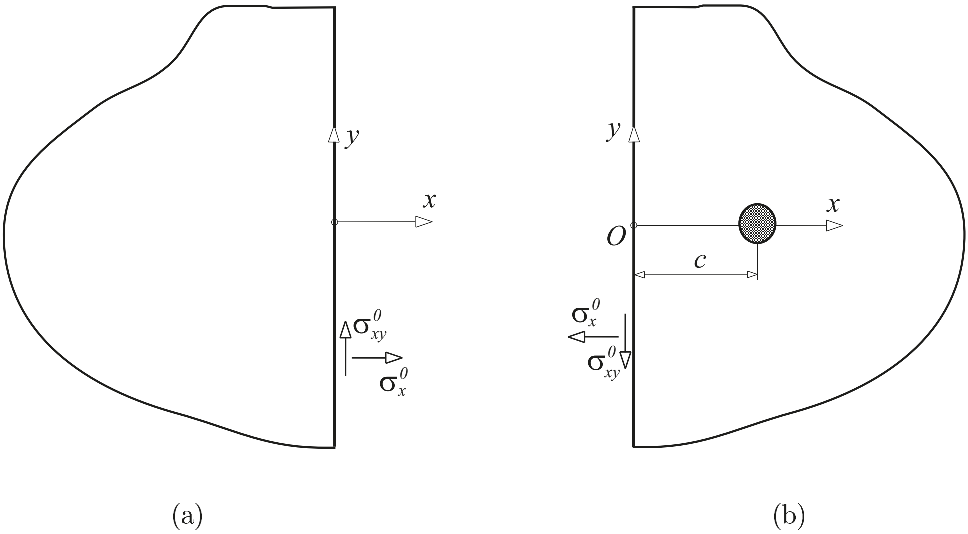


Figure 4: The free body diagrams of the left and right portions of the infinite medium from Fig. 3b. The infinite medium stresses along $x = 0$ are σ_x^0 and σ_{xy}^0.

To obtain the solution to the original problem from Fig. 3a, one needs to cancel the tractions along $x = 0$ in Fig. 4b, i.e., to superimpose to this problem the solution to the problem shown in Fig. 5a. The latter is related to the problem from Fig. 4a (left portion of the infinite medium problem) by the obvious symmetry/antisymmetry considerations, on the basis of which we can write

$$\hat{\sigma}_y(0, y) = -\sigma_y^{\sigma_0}(0, y) + \sigma_y^{\tau_0}(0, y)\,. \tag{3.2}$$

Therefore, by subtracting (3.1) from (3.2),

$$\hat{\sigma}_y(0, y) - \sigma_y^0(0, y) = -2\sigma_y^{\sigma_0}(0, y)\,, \quad \sigma_0 = \sigma_x^0(0, y)\,. \tag{3.3}$$

On the other hand, from the well-known result from two-dimensional elasticity, the longitudinal stress along the surface of a half-space due to distributed normal stress along that surface is exactly equal to that normal stress (Fig. 5b); see Timoshenko and Goodier (1970), eq. (g) on page 108, or Asaro and Lubarda (2006), eq. (12.23) on page 232. Thus,

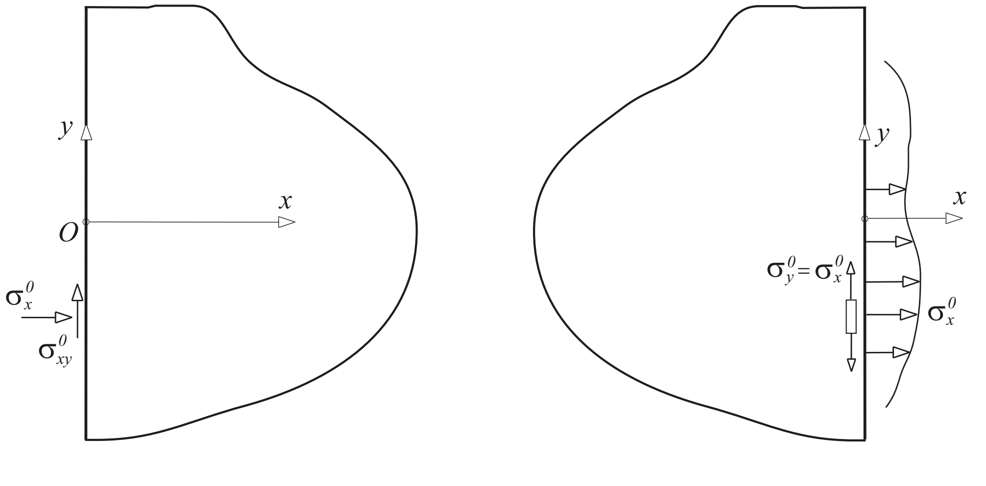
$$\sigma_y^{\sigma_0}(0, y) = \sigma_x^0(0, y)\,. \tag{3.4}$$

The substitution of (3.4) into (3.3) yields

$$\hat{\sigma}_y(0, y) = \sigma_y^0(0, y) - 2\sigma_x^0(0, y)\,. \tag{3.5}$$

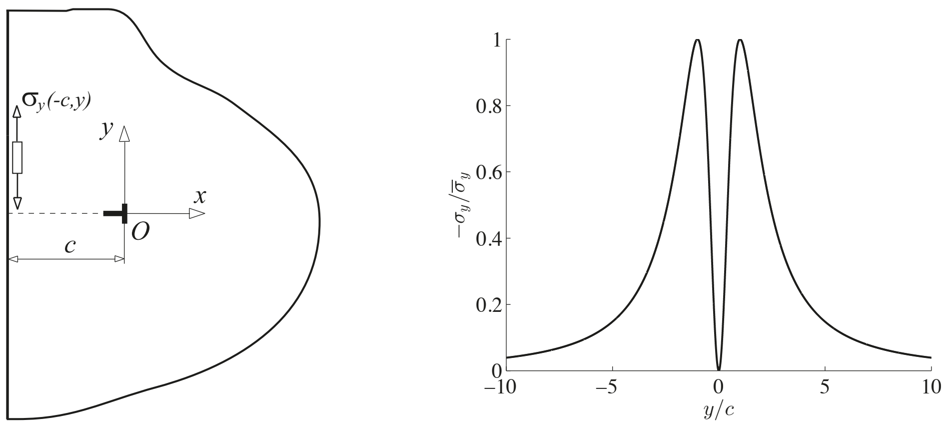
Since the longitudinal stress $\sigma_y(0, y)$ along the free surface of a half-space of the original problem from Fig. 3a is

$$\sigma_y(0, y) = \sigma_y^0(0, y) + \hat{\sigma}_y(0, y)\,, \tag{3.6}$$



(a) (b)

Figure 5: (a) To cancel the traction along $x = 0$ from Fig. 4b, the opposite traction is applied along the boundary of a half-space. (b) The longitudinal stress below the boundary $x = 0$ of a half-space, loaded by normal traction $\sigma_x^0(0, y)$, is equal to that traction, i.e., $\sigma_y^0(0, y) = \sigma_x^0(0, y)$.



(a) (b)

Figure 6: (a) An edge dislocation at distance c from the free surface of a half-space. (b) The variation of the stress $-\sigma_y(-c, y)$ with y/c. The scaling factor is $\bar{\sigma}_y = \mu(b_y/c)/[\pi(1 - \nu)]$.

by substituting (3.5) into (3.6) we deduce the desired formula

$$\sigma_y(0,y) = 2[\sigma_y^0(0,y) - \sigma_x^0(0,y)]\,. \tag{3.7}$$

4 Discussion

Among other uses, formula (3.7) can be conveniently applied to determine the longitudinal stress along the free surface of a half-space due to an edge dislocation beneath the free surface, without using the complete Head's (1953) solution to the edge dislocation in a half-space. Indeed, from the classical solution for the edge dislocation with the Burgers vector b_y in an infinite medium, the normal stresses are (Hirth and Lothe, 1982)

$$\sigma_x^0(x,y) = \frac{\mu b_y}{2\pi(1-\nu)}\,\frac{x(x^2-y^2)}{(x^2+y^2)^2}\,, \quad \sigma_y^0(x,y) = \frac{\mu b_y}{2\pi(1-\nu)}\,\frac{x(x^2+3y^2)}{(x^2+y^2)^2}\,, \tag{4.1}$$

with the coordinate origin at the center of the dislocation. Thus, if the free surface is at distance c from the dislocation, the longitudinal stress beneath the free surface (Fig. 6a) is obtained from (3.7) and (4.1) as

$$\sigma_y(-c,y) = -\bar{\sigma}_y\,\frac{4\eta^2}{(1+\eta^2)^2}\,, \quad \bar{\sigma}_y = \frac{\mu(b_y/c)}{\pi(1-\nu)}\,, \tag{4.2}$$

where $\eta = y/c$. The magnitude of the maximum stress in (4.2) occurs at $y = \pm c$, and is equal to $\bar{\sigma}_y$. While in an infinite medium $\sigma_y^0(-c,0) = -\bar{\sigma}_y/2$, for the dislocation in a half-space this stress relaxes to $\sigma_y(-c,0) = 0$ (Fig. 6b). The interaction between an edge dislocation and a circular void or inhomogeneity has been studied in great detail by Dundurs and Mura (1964), and Dundurs (1969). Their results have been used to address various problems in materials science. For example, the evaluation of the attraction exerted on a dislocation by the free surface of a nearby void plays a prominent role in the study of void growth by dislocation emission, which is a precursor to material failure by spalling under dynamic loadings (Lubarda and Meyers, 2003; Lubarda et al., 2004; Meyers et al., 2009; Rudd, 2009; Lubarda 2011a,b).

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